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STABILITY OF THE DETERMINATION OF A TIME-DEPENDENT COEFFICIENT IN PARABOLIC EQUATIONS

MOURAD CHOULLI AND YAVAR KIAN

Abstract. We establish a Lipschitz stability estimate for the inverse problem consisting in the determination of the coefficient $\sigma(t)$, appearing in a Dirichlet initial-boundary value problem for the parabolic equation $\partial_t u - \Delta_x u + \sigma(t)f(x)u = 0$, from Neumann boundary data. We extend this result to the same inverse problem when the previous linear parabolic equation is changed to the semi-linear parabolic equation $\partial_t u - \Delta_x u = F(t, x, \sigma(t), u(x,t))$.

Key words : parabolic equation, semi-linear parabolic equation, inverse problem, determination of time-depend coefficient, stability estimate.

AMS subject classifications : 35R30.

Contents

1. Introduction

Throughout this paper, we assume that Ω is a $\mathcal{C}^3$ bounded domain of $\mathbb{R}^n$ with $n \geqslant 2$. Let $T > 0$ and set

$$Q = \Omega \times (0,T), \quad \Gamma = \partial\Omega, \quad \Sigma = \Gamma \times (0,T).$$

We consider the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u + \sigma(t)f(x)u = 0, & (x,t) \in Q, \\ u(x,0) = h(x), & x \in \Omega, \\ u(x,t) = g(x,t), & (x,t) \in \Sigma. \end{cases} \tag{1.1}$$

We introduce the following assumptions :

(H1) $f \in \mathcal{C}^2(\overline{\Omega})$, $h \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, $g \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma})$, for some $0 < \alpha < 1$, and satisfy the compatibility condition

$$\partial_t g(x,0) - \Delta_x h(x) + \sigma(0)f(x)h(x) = 0, \quad x \in \Gamma.$$

(H2) There exists $x_0 \in \Gamma$ such that

$$\inf_{t \in [0,T]} |g(x_0,t)f(x_0)| > 0.$$

Under assumption (H1), it is well known that, for $\sigma \in \mathcal{C}^1[0,T]$, the initial-boundary value problem (1.1) admits a unique solution $u = u(\sigma) \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})$ (see Theorem 5.2 of [LSU]). Moreover, given $M > 0$, there exists a constant $C > 0$ depending only on data (that is Ω, T, f, g and h) such that $\|\sigma\|_{W^{1,\infty}(0,T)} \leqslant M$ implies

$$\|u(\sigma)\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})} \leqslant C. \tag{1.2}$$

In the present paper we are concerned with the inverse problem consisting in the determination of the time dependent coefficient $\sigma(t)$ from Neumann boundary data $\partial_\nu u(\sigma)$ on Σ, where ∂_ν is the derivative in the direction of the unit outward normal vector to Γ.

We prove the following theorem, where $B(M)$ is the ball of $C^1[0,T]$ centered at 0 and with radius $M>0$.

Theorem 1. *Assume that* (H1) *and* (H2) *are fulfilled. For $i=1,2$, let $\sigma_i \in B(M)$ and $u_i = u(\sigma_i)$. Then there exists a constant $C>0$, depending only on data, such that*

$$\|\sigma_2-\sigma_1\|_{L^\infty(0,T)} \leqslant C\,\|\partial_t\partial_\nu u_2-\partial_t\partial_\nu u_1\|_{L^\infty(\Sigma)}. \tag{1.3}$$

Following [COY], it is quite natural to extend Theorem 1 when the linear parabolic equation is changed to a semi-linear parabolic equation. To this end, introduce the following semi-linear initial-boundary value problem :

$$\begin{cases} \partial_t u-\Delta_x u = F(x,t,\sigma(t),u(x,t)), & (x,t)\in Q,\\ u(x,0)=h(x), & x\in\Omega,\\ u(x,t)=g(x,t), & (x,t)\in\Sigma \end{cases} \tag{1.4}$$

and consider the following assumptions

(H3) $h\in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, $g\in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Sigma})$, for some $0<\alpha<1$, and satisfy the compatibility condition

$$\partial_t g(x,0)-\Delta_x h(x) = F(0,x,\sigma(0),h(x)),\quad x\in\Gamma.$$

(H4) $F\in\mathcal{C}^1(\overline{\Omega}_x\times\mathbb{R}_t\times\mathbb{R}_\sigma\times\mathbb{R}_u)$ is such that $\partial_u F$ and $\partial_\sigma F$ are $\mathcal{C}^1$, F and $\partial_\sigma F$ are $\mathcal{C}^2$ with respect to x and u.

(H5) There exist $M>0$ and $x_0\in\Gamma$ such that

$$\inf_{t\in[0,T],\sigma\in[-M,M]} |\partial_\sigma F(x_0,t,\sigma,g(x_0,t))|>0.$$

(H6) There exist two non negative constants c and d such that

$$uF(x,t,\sigma(t),u)\leqslant cu^2+d,\quad t\in[0,T],\quad x\in\overline{\Omega},\quad u\in\mathbb{R}.$$

Under the above mentioned conditions, for any $\sigma\in\mathcal{C}^1[0,T]$, the initial-boundary value problem (1.4) admits a unique solution $u=u(\sigma)\in\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ (see Theorem 6.1 in [LSU]) and, given $M>0$, there exists a constant $C>0$ depending only on data (that is Ω, T, F, g and h) such that $\|\sigma\|_{W^{1,\infty}(0,T)}\leqslant M$ implies

$$\|u(\sigma)\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})}\leqslant C. \tag{1.5}$$

We have the following extension of Theorem 1.

Theorem 2. *Assume that* (H3), (H4), (H5) *and* (H6) *are fulfilled. For $i=1,2$, let $\sigma_i\in B(M)$ and $u_i=u(\sigma_i)$. Then there exists a constant $C>0$, depending only on data, such that*

$$\|\sigma_2-\sigma_1\|_{L^\infty(0,T)} \leqslant C\,\|\partial_t\partial_\nu u_2-\partial_t\partial_\nu u_1\|_{L^\infty(\Sigma)}. \tag{1.6}$$

Remark 1. *Let us observe that we can generalize the results in Theorems 1 and 2 as follows:*

i) In (1.1)*, we can replace $\sigma(t)f(x)$ by $\sum_{k=1}^p \sigma_k(t)f_k(x)$, where f_k, $1\leqslant k\leqslant p$, are known. Assume that $(H1)$ is satisfied, with $f=f_k$ for each k, where the compatibility condition is changed to*

$$\partial_t g(x,0)-\Delta_x h(x)+\sum_{k=1}^p \sigma_k(0)f_k(x)h(x)=0,\quad x\in\Gamma.$$

Therefore, to each $(\sigma_1,\dots,\sigma_p)\in\mathcal{C}[0,T]^p$ corresponds a unique solution $u=u(\sigma_1,\dots,\sigma_p)\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ and $\max\{\|\sigma_k\|_{W^{1,\infty}(0,T)};\ 1\leqslant k\leqslant p\}\leqslant M$ *implies*

$$\|u(\sigma_1,\dots,\sigma_p)\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})}\leqslant C,$$

for some positive constant C depending only on data.

Following the proof of Theorem 1, we prove that, under the following conditions : there exist $x_1, \ldots, x_p \in \Gamma$ *such that the matrix* $M(t) = (f_k(x_l)g(x_l,t))$ *is invertible for any* $t \in [0,T]$,

$$\max_{1\leqslant k\leqslant p} \|\sigma_k^1 - \sigma_k^2\|_{L^\infty(0,T)} \leqslant C \, \|\partial_t\partial_\nu u_2 - \partial_t\partial_\nu u_1\|_{L^\infty(\Sigma)} \, ,$$

if $\sigma_k^j \in B(M)$, $1 \leqslant k \leqslant p$ *and* $j = 1, 2$. *Here* C *is a constant that can depend only on data and* $u_j = u(\sigma_1^j, \ldots, \sigma_p^j)$, $j = 1, 2$.

ii) We can replace the semi-linear parabolic equation in (1.4) *by a semi-linear integro-differential equation. In other words,* F *can be changed to*

$$F_1(x,t,\sigma(t),u(x,t)) + \int_0^t F_2(x,s,\sigma(t-s),u(x,s))ds.$$

Under appropriate assumptions on F_1 *and* F_2, *one can establish that Theorem 2 is still valid in the present case.*

ii) Both in (1.1) *and* (1.4), *the Laplace operator can be replaced by a second order elliptic operator in divergence form :*

$$E = \nabla \cdot A(x)\nabla + B(x) \cdot \nabla,$$

where $A(x) = (a_{ij}(x))$ *is a symmetric matrix with coefficients in* $C^{1+\alpha}(\overline{\Omega})$, $B(x) = (b_i(x))$ *is a vector with components in* $C^\alpha(\overline{\Omega})$ *and the following ellipticity condition holds*

$$A(x)\xi \cdot \xi \geqslant \lambda \left|\xi\right|^2, \quad \xi \in \mathbb{R}^n, \ x \in \overline{\Omega}.$$

Actually, the normal derivative associated to E *is the boundary operator* $\partial_{\nu_E} = \nu(x) \cdot A(x)\nabla$.

To our knowledge, there are only few results concerning the determination of a time-dependent coefficient in an initial-boundary value problem for a parabolic equation from a single measurement. The determination of a source term of the form $f(t)\chi_D(x)$, where χ_D the characteristic function of the known subdomain D, was considered by J. R. Canon and S. P. Esteva. They established in [CE86-1] a logarithmic stability estimate in 1D case in a half line when the overdetermined data is the trace at the end point. A similar inverse problem problem in 3D case was studied by these authors in [CE86-2], where they obtained a Lipschitz stability estimate in weighted spaces of continuous functions. The case of a non local measurement was considered by J. R. Canon and Y. Lin in [CL88] and [CL90], where they proved existence and uniqueness for both quasilinear and semi-linear parabolic equations. The determination of a time dependent coefficient in an abstract integrodifferential equation was studied by the first author in [Ch91-1]. He proved existence, uniqueness and Lipschitz stability estimate, extending earlier results by [Ch91-2], [LS87], [LS88], [PO85-1] and [PO85-2]. In [CY06], the first author and M. Yamamoto obtained a stability result, in a restricted class, for the inverse problem of determining a source term $f(x,t)$, appearing in a Dirichlet initial-boundary value problem for the heat equation, from Neumann boundary data. In a recent work, the first author and M. Yamamoto [CY11] considered the inverse problem of finding a control parameter $p(t)$ that reach a desired temperature $h(t)$ along a curve $\gamma(t)$ for a parabolic semi-linear equation with homogeneous Neumann boundary data and they established existence, uniqueness as well as Lipschitz stability. Using geometric optic solutions, the first author [Ch09] proved uniqueness as well as stability for the inverse problem of determining a general time dependent coefficient of order zero for parabolic equations from Dirichlet to Neumann map. In [E07] and [E08], G. Eskin considered the same inverse problem for hyperbolic and the Schrödinger equations with time-dependent electric and magnetic potential and he established uniqueness by gauge invariance. Recently, R. Salazar [Sa] extended the result of [E07] and obtained a stability result for compactly supported coefficients.

We would like to mention that the determination of space dependent coefficient $f(x)$, in the source term $\sigma(t)f(x)$, from Neumann boundary data was already considered by the first author and M. Yamamoto [CY06]. But, it seems that our paper is the first work where one treats the determination of a time dependent coefficient, appearing in a parabolic initial-boundary value problem, from Neumann boundary data.

This paper is organized as follows. In section 2 we come back to the construction of the Neumann fundamental solution by [It] and establish time-differentiability of some potential-type functions, necessary for proving Theorems 1 and 2. Section 3 is devoted to the proof of Theorems 1 and 2.

2. Time-differentiability of potential-type functions

In this section, we establish time-differentiability of some potential-type functions, needed in the proof of our stability estimates. In our analysis we follow the construction of the fundamental solution by S. Itô [It].

First of all, we recall the definition of fundamental solution associate to the heat equation plus a time-dependent coefficient of order zero, in the case of Neumann boundary condition. Consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta_x u + q(x,t)u, & (x,t) \in \Omega \times (s,t_0), \\ \lim_{t\to s} u(x,t) = u_0(x), & x \in \Omega, \\ \partial_\nu u(x,t) = 0, & (x,t) \in \Gamma \times (s,t_0). \end{cases} \tag{2.1}$$

Here $s_0 < t_0$ are fixed, $s \in (s_0,t_0)$, u_0 and $q(x,t)$ are continuous respectively in $\overline{\Omega}$ and in $\overline{\Omega} \times [s,t_0]$. Let $U(x,t;y,s)$ be a continuous function in the domain $s_0 < s < t < t_0$, $x \in \overline{\Omega}$, $y \in \overline{\Omega}$. We recall that U is the fundamental solution of (2.1) if for any $u_0 \in \mathcal{C}(\overline{\Omega})$,

$$u(x,t) = \int_\Omega U(x,t;y,s)u_0(y)\mathrm{d}y$$

is the solution of (2.1). We refer to [It] for the existence and uniqueness of this fundamental solution.

We start with time-differentiability of volume potential-type functions[1].

Lemma 1. *Fix $s \in (s_0,t_0)$. Let $f \in \mathcal{C}(\overline{\Omega} \times [s,t_0])$ be $\mathcal{C}^2$ with respect to x, $q \in \mathcal{C}^1(\overline{\Omega} \times [s,t_0])$ and define, for $(x,t) \in \overline{\Omega} \times (s,t_0)$,*

$$f^1(x,t;\tau) = \int_\Omega U(x,t;y,\tau)f(y,\tau)\,dy, \quad t > \tau > s.$$

Then, f^1 admits a derivative with respect to t and

$$\begin{aligned} \frac{\partial f^1}{\partial t}(x,t;\tau) &= \int_\Omega U(x,t;y,\tau)(\Delta_y + q(x,\tau))f(y,\tau)\,dy \\ &\quad + \int_\tau^t \int_\Omega \int_\Omega U(x,t;z,\tau')\partial_t q(z,\tau')U(z,\tau';y,\tau)f(y,\tau)\,dz\,dy\,d\tau'. \end{aligned} \tag{2.2}$$

Moreover, F given by

$$F(x,t) = \int_s^t f^1(x,t;\tau)\,d\tau, \ (x,t) \in \Gamma \times (s_0,t_0),$$

possesses a derivative with respect to t,

$$\frac{\partial F}{\partial t}(x,t) = f(x,t) + \int_s^t \frac{\partial f^1}{\partial t}(x,t;\tau)\,d\tau \tag{2.3}$$

and

$$\left| \int_s^t \frac{\partial f^1}{\partial t}(x,t;\tau)\,d\tau \right| \leqslant C \int_s^t \|f(.,\tau)\|_{C_x^2(\overline{\Omega})}\,d\tau. \tag{2.4}$$

[1] Recall that if $\varphi = \varphi(x,t)$ is a continuous function then the corresponding volume potential is given by

$$\psi(x,t) = \int_s^t \int_\Omega U(x,t;y,\tau)\varphi(y,\tau)\mathrm{d}y\mathrm{d}\tau.$$

Proof. We have only to prove (2.2) and (2.4), because (2.3) follows immediately from (2.2).

Let then $u_0 \in \mathcal{C}^2(\overline{\Omega})$ and consider the function

$$u(x,t) = \int_\Omega U(x,t;y,s)u_0(y)\mathrm{d}y, \quad x \in \overline{\Omega},\ s < t < t_0.$$

We show that u admits a derivative with respect to t and

$$\begin{aligned}\partial_t u(x,t) &= \partial_t \left(\int_\Omega U(x,t;y,s)u_0(y)\mathrm{d}y \right) \\ &= \int_\Omega U(x,t;y,s)(\Delta_y + q(x,s))u_0(y)\mathrm{d}y \\ &\qquad - \int_s^t \int_\Omega \int_\Omega U(x,t;z,\tau)q_t(z,\tau)U(z,\tau;y,s)u_0(y)\mathrm{d}z\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{2.5}$$

We need to consider first the case $u_0 = w_0 \in \mathcal{C}^\infty(\overline{\Omega})$. Set

$$w(x,t) = \int_\Omega U(x,t;z,s)w_0(y)\mathrm{d}y, \quad x \in \overline{\Omega},\ s < t < t_0.$$

Clearly, $w(x,t)$ is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t w - \Delta_x w - q(x,t)w = 0, & (x,t) \in \Omega \times (s,t_0), \\ \lim_{t \to s} w(x,t) = w_0(x), & x \in \Omega, \\ \partial_\nu w(x,t) = 0, & (x,t) \in \Gamma \times (s,t_0) \end{cases}$$

and $w_1 = \partial_t w$ satisfies

$$\begin{cases} \partial_t w_1 - \Delta_x w_1 - q(x,t)w_1 = -\partial_t q w, & (x,t) \in \Omega \times (s,t_0), \\ \lim_{t \to s} w_1(x,t) = (\Delta_x + q(x,s))w_0(x), & x \in \Omega, \\ \partial_\nu w_1(x,t) = 0, & (x,t) \in \Gamma \times (s,t_0). \end{cases}$$

Therefore, (2.5), with w in place of u, is a consequence of Theorem 9.1 of [It].

Next, let $(w_0^n)_n$ be a sequence in $\mathcal{C}^\infty(\overline{\Omega})$ converging to u_0 in $C^2(\overline{\Omega})$ and $v(x,t)$ given by

$$\begin{aligned}v(x,t) &= \int_\Omega U(x,t;y,s)(\Delta_x + q(x,s))u_0(y)\mathrm{d}y \\ &\qquad - \int_s^t \int_\Omega \int_\Omega U(x,t;z,\tau)\partial_t q(z,\tau)U(z,\tau;y,s)u_0(y)\mathrm{d}z\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Consider $(w_n)_n$, the sequence of functions, defined by

$$w_n(x,t) = \int_\Omega U(x,t;z,s)w_0^n(y)\mathrm{d}y.$$

We proved that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\partial_t w_n(x,t) &= \int_\Omega U(x,t;y,s)(\Delta_y + q(x,s))w_0^n(y)\mathrm{d}y \\ &\qquad - \int_s^t \int_\Omega \int_\Omega U(x,t;z,\tau)\partial_t q(z,\tau)U(z,\tau;y,s)w_0^n(y)\mathrm{d}z\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{2.6}$$

From the proof of Theorem 7.1 of [It],

$$\int_\Omega |U(x,t;y,s)|\,\mathrm{d}y \leqslant Ce^{C(t-s)},\ (x,t) \in \overline{\Omega} \times (s,t_0). \tag{2.7}$$

Therefore, we can pass to the limit, as n goes to infinity, in (2.6). We deduce that $\partial_t w_n$ converges to v in $\mathcal{C}(\overline{\Omega} \times [s,t_0])$. But, w_n converges to u in $\mathcal{C}(\overline{\Omega} \times [s,t_0])$. Hence u admits a derivative with respect to t and

$\partial_t u = v$. That is we proved (2.5) and consequently (2.2) holds true. Finally, we note that (2.4) is deduced easily from (2.7). $\square$

Next, we consider time-differentiability a single layer potential-type function[2].

Lemma 2. *Fix $s \in (s_0, t_0)$. Let $f \in \mathcal{C}(\Gamma \times [s, t_0])$ be $\mathcal{C}^1$ with respect to $t \in [s, t_0]$ with $f(x, s) = 0$. Define, for $(x, t) \in \Gamma \times (s, t_0)$,*

$$f^1(x, t; \tau) = \int_\Gamma U(x, t; y, \tau) f(y, \tau) d\sigma(y), \quad t > \tau > s.$$

Then

$$F(x, t) = \int_s^t f^1(x, t; \tau) d\tau$$

is well defined, admits a derivative with respect to t and we have

$$\left\| \frac{\partial F}{\partial t} \right\|_{L^\infty(\Gamma \times (s, t_0))} \leqslant C \left\| \partial_t f \right\|_{L^\infty(\Gamma \times (s, t_0))}. \tag{2.8}$$

Contrary to Lemma 1, for Lemma 2 we cannot use directly the general properties of the fundamental solutions developed in [It]. We need to come back to the construction of the fundamental solution of (2.1) introduced by [It]. First, consider the heat equation $\partial_t u = \Delta_x u$ in the half space $\Omega_1 = \{x = (x_1, \ldots, x_n);\ x_1 > 0\}$ in $\mathbb{R}^n$ with the boundary condition $\partial_{x_1} u = 0$ on $\Gamma_1 = \{x = (0, x_2, \ldots, x_n);\ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\}$. For any $y = (y_1, y_2, \ldots, y_n)$, we define $\overline{y}$ by $\overline{y} = (-y_1, y_2, \ldots, y_n)$. Let

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{t}}$$

denotes the Gaussian kernel and set

$$G_1(x, t; y) = G(x - y, t) + G(x - \overline{y}, t).$$

Then, the fundamental solution $U_0(x, t; y, s)$ of

$$\begin{cases} \partial_t u = \Delta_x u, & (x, t) \in \Omega_1 \times (s, t_0), \\ \lim\limits_{t \to s} u(x, t) = u_0(x), & x \in \Omega_1, \\ \partial_\nu u(x, t) = 0, & (x, t) \in \Gamma_1 \times (s, t_0) \end{cases} \tag{2.9}$$

is given by

$$U_0(x, t; y, s) = G_1(x, t - s; y).$$

In order to construct the fundamental solution in the case of an arbitrary domain Ω, Itô introduced the following local coordinate system around each point $z \in \Gamma$.

Lemma 3. (Lemma 6.1 and its corollary, Chapter 6 of [It]) *For every point $z \in \Gamma$, there exist a coordinate neighborhood W_z of z and a coordinate system $(x_1^*, \ldots, x_n^*)$ satisfying the following conditions:*
1) the coordinate transformation between the coordinate system $(x_1^*, \ldots, x_n^*)$ and the original coordinate system in W_z is of class C^2 and the partial derivatives of the second order of the transformation functions are Hölder continuous ;
2) $\Gamma \cap W_z$ is represented by the equation $x_1^* = 0$ and $\Omega \cap W_z$ is represented by $x_1^* > 0$;
3) let $\mathcal{L}$ be the diffeomorphism from W_z to $\mathcal{L}(W_z)$ defined by

$$\begin{array}{rccc} \mathcal{L}: & W_z & \to & \mathcal{L}(W_z) \\ & x & \mapsto & (x_1^*(x), \ldots, x_n^*(x)). \end{array}$$

[2]The single-layer potential corresponding to a continuous function $\varphi = \varphi(x, t)$ is given by

$$\psi(x, t) = \int_s^t \int_\Gamma U(x, t; y, \tau) \varphi(y, \tau) \mathrm{d}\sigma(y) \mathrm{d}\tau.$$

Then, for any $u \in \mathcal{C}^1(\overline{\Omega})$ we have

$$\partial_\nu u(\xi) = -\partial_{x_1}(u \circ \mathcal{L}^{-1})(x), \quad \xi \in \Gamma \cap W_z \quad \textit{and} \quad x = \mathcal{L}(\xi).$$

From now on, for any $z \in \Gamma$, we view coordinate system $(x_1^*, \dots, x_n^*)$ as a rectangular coordinate system. Moreover, using the local coordinate system of Lemma 3, for any $y = (y_1, y_2, \dots, y_n) \in \mathcal{L}(W_z)$, we define $\overline{y} = (-y_1, y_2, \dots, y_n)$ and, without loss of generality, we assume that, for any $y \in \mathcal{L}(W_z)$, we have $\overline{y} \in \mathcal{L}(W_z)$. For any interior point z of Ω, we fix an arbitrary local coordinate system and a coordinate neighborhood W_z contained in Ω. For any $z \in \overline{\Omega}$ and $\delta > 0$, we set $W(z, \delta) = \{x : \ |x - z|^2 < \delta\}$ and $\delta_z > 0$ such that, for any $z \in \overline{\Omega}$ we have $\overline{W(z, \delta_z)} \subset W_z$.

Recall the following partition of unity lemma.

Lemma 4. (Lemma 7.1, Chapter 7 of [It]) *There exist a finite subset $\{z_1, \dots, z_m\}$ of $\overline{\Omega}$ and a finite sequence of functions $\{\omega_1, \dots, \omega_m\}$ with the following properties:*
1) supp $\omega_l \subset W(z_l, \delta_{z_l})$, $l = 1, \dots, m$, and each ω_l is of class $\mathcal{C}^3$ with respect to the local coordinates in W_{z_l} ;
2) $\{\omega_l(x)^2;\ l = 1, \dots, m\}$ forms a partition of unity in $\overline{\Omega}$;
3)$\partial_\nu \omega_l(\xi) = 0$, $l = 1, \dots, m$, $\xi \in \Gamma$.

Let $\{z_1, \dots, z_m\}$ be the finite subset of $\overline{\Omega}$, introduced in the previous lemma. For any $k \in \{1, \dots, m\}$, let $\mathcal{L}_k$ denotes the diffeomorphism from W_{z_k} to $\mathcal{L}_k(W_{z_k})$ defined by

$$\begin{array}{rccc} \mathcal{L}_k : & W_{z_k} & \to & \mathcal{L}_k(W_{z_k}) \\ \\ & x & \mapsto & (x_1^*(x), \dots, x_n^*(x)), \end{array}$$

where $(x_1^*, \dots, x_n^*)$ is the local coordinate system of Lemma 3 defined in W_{z_k}. For any $k \in \{1, \dots, m\}$, the differential operator

$$\partial_t - \Delta_x - q(x,t)$$

becomes, in terms of local coordinate system $x^* = (x_1^*, \dots, x_n^*)$,

$$L_{t,x^*}^k = \partial_t - \frac{1}{\sqrt{a_k(x^*)}} \sum_{i,j=1}^n \partial_{x_i^*}\left(\sqrt{a_k(x^*)}a_k^{ij}(x^*)\partial_{x_j^*}\cdot\right) - q_k(x^*, t)$$

in $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$. Here $q_k(x^*, t)$ is Hölder continuous on $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$ and $(a^{ij}(x^*))$ is the contravariant tensor of degree 2 defined by

$$\left(a_k^{ij}(x^*)\right) = \left(J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*))\right)^T \left(J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*))\right),$$

with

$$J_{\mathcal{L}_k}(x) = \left(\frac{\partial x_j^*(x)}{\partial x_i}\right).$$

According to the construction of [It] given in Chapter 6 (see pages 42 to 45 of [It]), $\left(a_k^{ij}(x^*)\right)$ is of class $\mathcal{C}^2$ in $\mathcal{L}_k(\overline{\Omega} \cap W_{z_k})$ and it is a positive definite symmetric matrix at every point $x^* \in \mathcal{L}_k(W_{z_k})$. We set $(a_{ij}^k(x^*)) = (a_k^{ij}(x^*))^{-1}$ and $a_k(x^*) = \det(\mathrm{a}_{\mathrm{ij}}^{\mathrm{k}}(\mathrm{x}^*))$. Consider the volume element $\mathrm{d}x^* = \sqrt{a_k(x^*)}\mathrm{d}x_1^* \dots \mathrm{d}x_n^*$ on $\mathcal{L}_k(W_{z_k})$ and $\mathrm{d}x' = \sqrt{a(0, x')}\mathrm{d}x_2^* \dots \mathrm{d}x_n^*$ on $\mathcal{L}_k(W_{z_k} \cap \Gamma)$ with $x' = (x_2^*, \dots, x_n^*)$. Note that, by the construction of S. Itô [It] (see page 45), for any $k = 1, \dots, m$, we have

$$a_k^{ij}(\mathcal{L}_k(x)) = a_k^{ij}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } i = j = 1 \text{ or } i, j = 2, \dots, n, \tag{2.10}$$

$$a_k^{1j}(\mathcal{L}_k(x)) = a_k^{j1}(\mathcal{L}_k(x)) = -a_k^{1j}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } j = 1, \dots, n \tag{2.11}$$

and

$$a_k^{1j}(\mathcal{L}_k(\xi)) = a_k^{j1}(\mathcal{L}_k(\xi)) = \delta_{j1}, \quad \xi \in \Gamma \cap W_{z_k}, \quad j = 1, \dots, n, \tag{2.12}$$

where δ_{j1} denotes the kronecker's symbol. For any $k \in \{1, \dots, m\}$, let $G_k(x, t; y)$ be defined, in the region

$$D_k = \{(x, t, y);\ x,\ y \in \mathcal{L}_k(W_{z_k}),\ 0 < t < t_0 - s\},$$

by

$$G_k(x,t;y)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\sum_{i,j=1}^n \frac{a_{ij}^k(y)(x_i-y_i)(x_j-y_j)}{4t}}.$$

Next , define $H_{z_k}(x,t;y) = G_k(\mathcal{L}_k(x),t;\mathcal{L}_k(y))$, for $k\in\{1,\ldots,m\}$ and $z_k\in\Omega$; $H_{z_k}(x,t;y) = G_k(\mathcal{L}_k(x),t;\mathcal{L}_k(y))+G_k(\mathcal{L}_k(x),t;\overline{\mathcal{L}_k(y)})$, for $k\in\{1,\ldots,m\}$, $z_k\in\Gamma$, $x\in W_{z_k}$ and $y\in W_{z_k}$; $H_{z_k}(x,t;y)=0$ if $x\notin W_{z_k}$ or $y\notin W_{z_k}$. Consider also $H(x,t;y)$, defined in the region

$$D=\{(x,t,y);\ x\in\overline{\Omega},\ y\in\overline{\Omega},\ 0<t<t_0-s\},$$

as follows

$$H(x,t;y)=\sum_{l=1}^m \omega_l(x)H_{z_l}(x,t;y)\omega_l(y).$$

As in Lemma 7.2 of [It], we define successively:

$$J_0(x,t;y,s)=(\partial_t-\Delta_x-q(x,t))(H(x,t-s;y)),$$

$$J_k(x,t;y,s)=\int_s^t\int_\Omega J_0(x,t;z,\tau)J_{k-1}(z,\tau;y,s)\mathrm{d}z\mathrm{d}\tau,$$

$$K(x,t;y,s)=\sum_{k=0}^{+\infty}J_k(x,t;y,s).$$

Then, following [It] (see page 53), the fundamental solution of (2.1) is given by

$$U(x,t;s,y)=H(x,t-s;y)+\int_s^t\int_\Omega H(x,t-\tau;z)K(z,\tau;y,s)\mathrm{d}z\mathrm{d}\tau. \tag{2.13}$$

We are now able to prove Lemma 2 with the help of representation (2.13), the properties of $H(x,t;y)$ and $K(x,t;y,s)$.

Proof of Lemma 2. Without loss of generality, we assume that $s=0$. Set

$$F_1(x,t)=\int_0^t\int_\Gamma H(x,t-s;y)f(y,s)\mathrm{d}\sigma(y)\mathrm{d}s,$$

$$F_2(x,t)=\int_0^t\int_\Gamma\int_s^t\int_\Omega H(x,t-\tau;z)K(z,\tau;y,s)f(y,s)\mathrm{d}z\mathrm{d}\tau\mathrm{d}\sigma(y)\mathrm{d}s.$$

According to representation (2.13), one needs to show that F_1 and F_2 admit a derivative with respect to t and

$$|\partial_t F_1(x,t)|+|\partial_t F_2(x,t)|\leqslant C\,\|\partial_t f\|_{L^\infty(\Gamma\times(0,t_0))} \tag{2.14}$$

for $(x,t)\in\Gamma\times(0,t_0)$. We start by considering F_1. Applying a simple substitution, we obtain

$$F_1(x,t)=\int_0^t\int_\Gamma H(x,s;y)f(y,t-s)\mathrm{d}\sigma(y)\mathrm{d}s. \tag{2.15}$$

Next, for $x\in\Gamma$, there exist $l_1,\ldots,l_r\subset\{1,\ldots,m\}$ such that $x\in\operatorname{supp}\omega_l$ for $l\in\{l_1,\ldots,l_r\}$ and $x\notin\operatorname{supp}\omega_l$ for $l\notin\{l_1,\ldots,l_r\}$. Moreover, since $x\in\Gamma$, we have $z_{l_1},\ldots,z_{l_r}\in\Gamma$. Then, from the construction of $H(x,t;y)$, we obtain

$$\begin{aligned}\int_\Gamma H(x,s;y)\mathrm{d}\sigma(y)&=\int_\Gamma\sum_{k=1}^r\omega_{l_k}(x)H_{z_{l_k}}(x,s;y)\omega_{l_k}(y)\mathrm{d}\sigma(y)\\&=2\sum_{k=1}^r\int_{\mathbb{R}^{n-1}}\chi_{l_k}(0,x')\frac{1}{(4\pi s)^{\frac{n}{2}}}e^{-\sum_{i,j=1}^n\frac{a_{ij}^{l_k}(0,y')(x_i'-y_i')(x_j'-y_j')}{4s}}\chi_{l_k}(0,y')\sqrt{a_{l_k}(0,y')}\mathrm{d}y'\end{aligned}$$

with, for $l\in\{1,\ldots,m\}$, $\chi_l\in\mathcal{C}_0^3\left(\mathcal{L}_l(\operatorname{supp}\omega_l)\right)$ such that $\chi_l(x)=\omega_l(\mathcal{L}_l^{-1}(x))$ and with $(x_1',\ldots,x_n')=(0,x')$, $(y_1',\ldots,y_n')=(0,y')$. Using the substitution $y'\to z'=\frac{x'-y'}{\sqrt{s}}$, we derive

$$\begin{aligned}&\int_\Gamma H(x,s;y)\mathrm{d}\sigma(y)\\&\leqslant C\sum_{k=1}^r\int_{\mathbb{R}^{n-1}}\chi_{l_k}(0,x')\frac{1}{\sqrt{s}}e^{-\sum_{i,j=1}^n a_{ij}^{l_k}(0,x'-\sqrt{s}z')z_i'z_j'}\chi_{l_k}(0,x'-\sqrt{s}z')\sqrt{a_{l_k}(0,x'-\sqrt{s}z')}\mathrm{d}z'.\end{aligned}\tag{2.16}$$

Therefore,

$$\int_\Gamma |H(x,s;y)|\,\mathrm{d}\sigma(y)\leqslant\frac{C}{\sqrt{s}}\int_{\mathbb{R}^{n-1}}e^{-a_0|z'|^2}dz'\leqslant\frac{C}{\sqrt{s}},$$

where $a_0>0$ is a constant. From this estimate, we deduce that

$$\int_\Gamma |H(x,s;y)f(y,t-s)|\,\mathrm{d}\sigma(y)\leqslant C\frac{\|f\|_{L^\infty(\Gamma\times(0,t_0))}}{\sqrt{s}}$$

and

$$\left|\partial_t\left(\int_\Gamma H(x,s;y)f(y,t-s)\mathrm{d}\sigma(y)\right)\right|\leqslant C\frac{\|\partial_t f\|_{L^\infty(\Gamma\times(0,t_0))}}{\sqrt{s}}.$$

Thus, F_1 admits a derivative with respect to t,

$$\partial_t F_1(x,t)=\int_\Gamma H(x,t;y)f(y,0)\mathrm{d}\sigma(y)+\int_0^t\int_\Gamma H(x,s;y)\partial_t f(y,t-s)\mathrm{d}\sigma(y)\mathrm{d}s$$

and, since $f(y,0)=0$ for $y\in\Gamma$, we obtain

$$|\partial_t F_1(x,t)|\leqslant C\left\|\partial_t f\right\|_{L^\infty(\Gamma\times(0,t_0))},\quad (x,t)\in\Gamma\times(0,t_0).\tag{2.17}$$

Let us now consider F_2. We want to show that $\partial_t F_2$ exists and the following estimate holds:

$$|\partial_t F_2(x,t)|\leqslant C\left\|\partial_t f\right\|_{L^\infty(\Gamma\times(0,t_0))},\quad (x,t)\in\Gamma\times(0,t_0).\tag{2.18}$$

For this purpose, using the local coordinate system, it suffices to prove

$$\left|\partial_t F_2(\mathcal{L}_l^{-1}(0,x'),t)\right|\leqslant C_l\left\|\partial_t f\right\|_{L^\infty((\Gamma\times(0,t_0))},\quad ((0,x'),t)\in\mathcal{L}_l(\Gamma\cap W_{z_l})\times(0,t_0),\quad l\in\{1,\ldots,m\}.\tag{2.19}$$

From now on we set $x=\mathcal{L}_l^{-1}(0,x')$ with $(0,x')\in\mathcal{L}_l(\Gamma\cap W_{z_l})\subset\{0\}\times\mathbb{R}^{n-1}$ and we will show (2.19). First, note that

$$\begin{aligned}J_0(z,\tau;s,y)=&(\partial_\tau-\Delta_z-q(z,t))H(z,\tau-s;y)\\=&\sum_{l=1}^m\omega_l\left(\mathcal{L}_l^{-1}(z^*)\right)L^l_{\tau,z^*}H_{z_l}(\mathcal{L}_l^{-1}(z^*),\tau-s;y)\omega_l(y)\\&+\sum_{l=1}^m[L^l_{\tau,z^*},\omega_l(\mathcal{L}_l^{-1}(z^*))]H_{z_l}(\mathcal{L}_l^{-1}(z^*),\tau-s;y)\omega_l(y).\end{aligned}$$

According to the results in Chapter 4 of [It] (pages 26 and 27), using the local coordinate system, we obtain

$$\begin{aligned}L^l_{\tau,z^*}H_{z_l}(\mathcal{L}_l^{-1}(z^*),\tau-s;\mathcal{L}_l^{-1}(y^*))=&\sum_{i,j=1}^n(a_l^{ij}(z^*)-a_l^{ij}(y^*))\frac{\partial^2 H_{z_l}}{\partial_{z_i^*}\partial_{z_j^*}}(\mathcal{L}_l^{-1}(z^*),\tau-s;\mathcal{L}_l^{-1}(y^*))\\&+[B_l(z^*,y^*,\partial_{z^*})+q_l(z^*,t)]H_{z_l}(\mathcal{L}_l^{-1}(z^*),\tau-s;\mathcal{L}_l^{-1}(y^*)),\end{aligned}$$

where $B_l(z^*,y^*,\partial_{z^*})$ is a differential operator of order $\leqslant 1$ in z^* with continuous coefficients in $z^*,y^*\in\mathcal{L}_l(\operatorname{supp}\omega_l)$. In view of the results in Chapter 4 of [It] (see pages 26 and 27), combining (2.10), (2.11), (2.12) and (2.16), applying the substitution $y''=\frac{z'-y'}{\sqrt{\tau-s}}$, with $z^*=(z_1^*,z')$ and $y^*=(0,y')$, we obtain

$$\int_\Gamma J_0(\mathcal{L}_l^{-1}(z^*),\tau;y,s)\mathrm{d}\sigma(y)=\sum_{j=0}^2 P_j\left(\frac{z_1^*}{\sqrt{\tau-s}}\right)e^{-\frac{(z_1^*)^2}{\tau-s}}\left[\int_{\mathbb{R}^{n-1}}\frac{J_0^j\left(z^*,\tau;y'',s;\tau-s\right)}{(\tau-s)^{\frac{j}{2}}}\mathrm{d}y''\right],$$

for $0<s<\tau<t_0$ and $z^*\in\mathcal{L}_l(\Omega\cap W_{z_k})$, where, for $j=0,1,2$, P_j are polynomials and J_0^j are continuous functions, $\mathcal{C}^1$ with respect to τ, $s\in(0,t_0)$ and satisfy

$$\max_{\substack{i=0,1,2\\ \alpha_1+\alpha_2\leqslant 1}}\int_{\mathbb{R}^{n-1}}\left|\partial_\tau^{\alpha_1}\partial_s^{\alpha_2}J_0^j\left(z^*,\tau;y'',s;v_1\right)\right|\mathrm{d}y''\leqslant C_l,\quad 0<s<\tau<t_0,\quad z_1^*>0,\quad 0<\ v_1<t_0,$$

for some constant $C_l>0$. We note that $\partial_{v_1}J_0^j\left((z_1',z''),\tau;y'',s;v_1\right)$ is not necessarily bounded. Indeed, we show

$$\left|\partial_{v_1}J_0^j\left((z_1',z''),\tau;y'',s;v_1\right)\right|\leqslant\frac{C_l}{\sqrt{v_1}},\quad 0<\ v_1<t_0,\quad j=0,1,2.$$

This representation and the construction of $K(z,\tau;y,s)$ in Chapter 5 of [It] (see pages 31 to 32 for the construction in $\mathbb{R}^n$ and page 53 for the construction in a bounded domain) lead

$$\int_\Gamma K(\mathcal{L}_l^{-1}(z^*),\tau;y,s)\mathrm{d}\sigma(y)=\sum_{j=0}^2 Q_j\left(\frac{z_1^*}{\sqrt{\tau-s}}\right)e^{-\frac{(z_1^*)^2}{\tau-s}}\left[\int_{\mathbb{R}^{n-1}}\frac{K_j\left(z^*,\tau;y'',s;\tau-s\right)}{(\tau-s)^{\frac{j}{2}}}\mathrm{d}y''\right],\tag{2.20}$$

for $0<s<\tau<t_0$ and $z^*\in\mathcal{L}_l(\Omega\cap W_{z_k})$, where, for $j=0,1,2$, Q_j are polynomials and K_j are continuous functions, $\mathcal{C}^1$ with respect to τ, $s\in(0,t_0)$ and satisfy

$$\max_{\substack{i=0,1,2\\ \alpha_1+\alpha_2\leqslant 1}}\int_{\mathbb{R}^{n-1}}\left|\partial_\tau^{\alpha_1}\partial_s^{\alpha_2}K_j\left(z^*,\tau;y'',s;v_1\right)\right|\mathrm{d}y''\leqslant C_l,\quad 0<s<\tau<t_0,\quad z_1^*>0,\quad 0<\ v_1<t_0,$$

where $C_l>0$ is a constant. Furthermore, using representation (2.20), we have, for $s<\tau<t<t_0$,

$$\begin{aligned}&\int_\Omega H(x,t-\tau;z)\int_\Gamma K(\tau,z;y,s)f(y,s)\mathrm{d}\sigma(y)\mathrm{d}z\\&=\sum_{j=0}^2\sum_{l=1}^m\int_{\mathbb{R}^n_+}\omega_l(x)H_{z_l}(x,t-\tau;\mathcal{L}_l^{-1}(z^*))\chi_l(z^*)Q_j\left(\frac{z_1^*}{\sqrt{\tau-s}}\right)e^{-\frac{(z_1^*)^2}{\tau-s}}\left[\int_{\mathbb{R}^{n-1}}\frac{K_j\left(z^*,\tau;y'',s;\tau-s\right)}{(\tau-s)^{\frac{j}{2}}}\mathrm{d}y''\right]\mathrm{d}z^*\end{aligned}$$

with $\mathbb{R}^n_+=\{(z_1^*,\ldots,z_n^*)\in\mathbb{R}^n;\ z_1^*>0\}$. Then, applying the substitutions $z''=\frac{x'-z'}{\sqrt{t-\tau}}$ and $z_1'=\frac{z_1^*}{\sqrt{\tau-s}}$, we deduce, in view of the form of the functions K_j, the following

$$\begin{aligned}&\int_\Omega H(x,t-\tau;z)\int_\Gamma K(z,\tau;y,s)\mathrm{d}\sigma(y)\mathrm{d}z\\&=\sum_{j=0}^1\int_{\mathbb{R}^n_+}\frac{H_l'(x',t-\tau;(z_1',z''),\tau-s)}{\sqrt{t-\tau}}\left[\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),\tau;y'',s;\tau-s\right)}{(\tau-s)^{\frac{j}{2}}}\mathrm{d}y''\right]\mathrm{d}z''\mathrm{d}z_1',\quad s<\tau<t<t_0,\end{aligned}\tag{2.21}$$

for some continuous functions K_0', K_1' and H_l' such that K_0', K_1' are $\mathcal{C}^1$, with respect to s and τ, and the following estimates hold:

$$\int_{\mathbb{R}^n_+}\left|H_l'(x',t-\tau;(z_1',z''),\tau-s)\right|\mathrm{d}z''\leqslant C_l,\quad 0<s<\tau<t<t_0,\tag{2.22}$$

$$\max_{\substack{j=0,1\\ \alpha_1+\alpha_2\leqslant 1}}\int_{\mathbb{R}^{n-1}}\left|\partial_\tau^{\alpha_1}\partial_s^{\alpha_2}K_j'\left((z_1',z''),\tau;y'',s;v_1\right)\right|\mathrm{d}y''\leqslant C_l,\quad 0<s<\tau<t_0,\quad 0<\ v_1<t_0,\tag{2.23}$$

for some constant $C_l > 0$. Repeating the arguments used for (2.21) and applying some results of page 31 of [It], we obtain, for $0 < t < t_0$,

$$\begin{aligned}\int_0^t\int_s^t\int_\Omega |H(x,t-\tau;z)|\int_\Gamma |K(z,\tau;y,s)|\,\mathrm{d}\sigma(y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s &\leqslant C_l\int_0^t\int_s^t\left[\sum_{j=0}^1\frac{1}{\sqrt{t-\tau}}\cdot\frac{1}{(\tau-s)^{\frac{j}{2}}}\right]\mathrm{d}\tau\mathrm{d}s\\ &\leqslant C_l\sum_{j=0}^1\int_0^t(t-s)^{1-\frac{j}{2}}\mathrm{d}s\leqslant C_l.\end{aligned}\tag{2.24}$$

This estimate and Fubini's theorem imply

$$F_2(x,t)=\int_0^t\int_s^t\int_\Omega H(x,t-\tau;z)\int_\Gamma K(z,\tau;y,s)f(y,s)\mathrm{d}\sigma(y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s.$$

Then, in view of representation (2.21), for all $0<t<t_0$,

$$\begin{aligned}F_2(x,t)=&\int_0^t\int_s^t\sum_{j=0}^1\int_{\mathbb{R}^n_+}\frac{H_l'(x',t-\tau;(z_1',z''),\tau-s)}{\sqrt{t-\tau}}\\ &\times\left[\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),\tau;y'',s;\tau-s\right)}{(\tau-s)^{\frac{j}{2}}}f_1(x',s;y'',z'')\mathrm{d}y''\right]\mathrm{d}z''\mathrm{d}z_1'\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

where $f_1(x',s;y'',z'')=f\left(\mathcal{L}_l^{-1}(0,x'-(\sqrt{t-s})z''-(\sqrt{\tau-s})y''),s\right)$. Making the substitution $\tau'=t-\tau$, we obtain

$$\begin{aligned}F_2(t,x)=&\int_0^t\int_0^{t-s}\sum_{j=0}^1\int_{\mathbb{R}^n_+}\frac{H_l'(x',\tau';(z_1',z''),t-s-\tau')}{\sqrt{\tau'}}\\ &\times\left[\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),t-\tau';y'',s;t-s-\tau'\right)}{(t-s-\tau')^{\frac{j}{2}}}f_1(x',s;y'',z'')\mathrm{d}y''\right]\mathrm{d}z''\mathrm{d}z_1'\mathrm{d}\tau'\mathrm{d}s.\end{aligned}$$

Then, the substitution $s'=t-s$ yields

$$\begin{aligned}F_2(x,t)=&\int_0^t\int_0^{s'}\sum_{j=0}^1\int_{\mathbb{R}^n_+}\frac{H_l'(x',\tau';(z_1',z''),s'-\tau')}{\sqrt{\tau'}}\\ &\times\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),t-\tau';y'',t-s';s'-\tau'\right)}{(s'-\tau')^{\frac{j}{2}}}f_1(x',t-s';y'',z'')\mathrm{d}y''\mathrm{d}z_1'\mathrm{d}z''\mathrm{d}\tau'\mathrm{d}s'.\end{aligned}$$

But, for $0<\tau'<s'<t<t_0$, estimates (2.22), (2.23) and $f(y,0)=0$, $y\in\Gamma$, imply

$$\begin{aligned}\Big|\sum_{j=0}^1\int_{\mathbb{R}^n_+}&\frac{H_l'(x',\tau',x';(z_1',z''),s'-\tau')}{\sqrt{\tau'}}\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),t-\tau';y'',t-s';s'-\tau'\right)}{(s'-\tau')^{\frac{j}{2}}}\\ &\times f_1(x',t-s';y'',z'')\mathrm{d}y''\mathrm{d}z''\mathrm{d}z_1'\Big|\leqslant C_l\sum_{j=0}^1\frac{\|\partial_t f\|_{L^\infty(\Gamma\times(0,t_0))}}{\sqrt{\tau'}}\frac{1}{(s'-\tau')^{\frac{j}{2}}}\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}\Big|\partial_t\Big(\sum_{j=0}^1\int_{\mathbb{R}^n_+}&\frac{H_l'(x',\tau';(z_1',z''),s'-\tau')}{\sqrt{\tau'}}\int_{\mathbb{R}^{n-1}}\frac{K_j'\left((z_1',z''),t-\tau';y'',t-s';s'-\tau'\right)}{(s'-\tau')^{\frac{j}{2}}}\\ &\times f_1(x',t-s';y'',z'')\mathrm{d}y''\mathrm{d}z_1'\mathrm{d}z''\Big)\Big|\leqslant C_l\sum_{j=0}^1\frac{\|\partial_t f\|_{L^\infty(\Gamma\times(0,t_0))}}{\sqrt{\tau'}}\frac{1}{(s'-\tau')^{\frac{j}{2}}}.\end{aligned}\tag{2.26}$$

From estimates (2.25), (2.26) and $f(y,0)=0$, $y\in\Gamma$, we conclude that F_2 admits a derivative with respect to t and

$$\partial_t F_2(x,t)=\int_0^t\int_0^{s'}\partial_t\Big(\sum_{j=0}^{1}\int_{\mathbb{R}^n_+}\frac{H_l'(x',\tau';(z_1',z''),s'-\tau')}{\sqrt{\tau'}}\int_{\mathbb{R}^{n-1}}\frac{K_j'((z_1',z''),t-\tau';y'',t-s';s'-\tau')}{(s'-\tau')^{\frac{j}{2}}}$$
$$\times f_1(x',t-s';y'';z'')\mathrm{d}y''\mathrm{d}z_1'\mathrm{d}z''\Big)\mathrm{d}\tau'\mathrm{d}s'.$$

Moreover, (2.24) and (2.26) imply (2.19) and (2.18). Finally, we obtain (2.14) from (2.17) and (2.18). This completes the proof. □

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Let $u=u_1-u_2$ and $\sigma=\sigma_2-\sigma_1$. Then u is the solution of the following initial-boundary value problem

$$\begin{cases}\partial_t u-\Delta_x u+\sigma_2(t)f(x)u=\sigma(t)f(x)u_1(x,t), & (x,t)\in Q,\\ u(x,0)=0, & x\in\Omega,\\ u(x,t)=0, & (x,t)\in\Sigma.\end{cases}\tag{3.1}$$

Let $U(x,t;y,s)$ be the fundamental solution of (2.1) with $q(x,t)=-\sigma_2(t)f(x)$. Applying Theorem 9.1 of [It], we obtain

$$u(x,t)=\int_0^t\int_\Omega U(x,t;y,s)\sigma(s)f(y)u_1(y,s)\mathrm{d}y\mathrm{d}s+\int_0^t\int_\Gamma U(x,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s.\tag{3.2}$$

Now, since $u(x,t)=0$, $(t,x)\in\Sigma$ and $x\in\Gamma$,

$$\int_0^t\int_\Omega U(x,t;y,s)\sigma(s)f(y)u_1(y,s)\mathrm{d}y\mathrm{d}s=-\int_0^t\int_\Gamma U(x,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s.\tag{3.3}$$

In view of differentiability properties in Lemma 1 and 2, we can take the t-derivative of both sides of identity (3.2). We find

$$f(x)g(x,t)\sigma(t)=-\int_0^t\partial_t\left(\int_\Omega U(x,t;y,s)\sigma(s)f(y)u_1(y,s)\mathrm{d}y\right)\mathrm{d}s$$
$$-\partial_t\left(\int_0^t\int_\Gamma U(x,t;y,s)(\partial_\nu u_2(y,s)-\partial_\nu u_1(y,s))\mathrm{d}\sigma(y)\mathrm{d}s\right)$$

and, for $x=x_0$, condition (H2) implies

$$\begin{aligned}\sigma(t)=h(t)\int_0^t\partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)f(y)u_1(y,s)\mathrm{d}y\right)\mathrm{d}s\\ +h(t)\partial_t\left(\int_0^t\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s\right),\end{aligned}\tag{3.4}$$

where $h(t)=-1/(g(t,x_0)f(x_0))$.

Since $u(x,0)=0$, $x\in\Omega$, we have $\partial_\nu u(x,0)=0$, $x\in\Gamma$. Thus, the estimates in Lemma 1 and 2 lead

$$\left|\int_0^t\partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)f(y)u_1(y,s)\mathrm{d}y\right)\mathrm{d}s\right|\leqslant C\int_0^t|\sigma(s)|\,\mathrm{d}s,\tag{3.5}$$

$$\left|\partial_t\left(\int_0^t\int_\Gamma U(x,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s\right)\right|\leqslant C\,\|\partial_t\partial_\nu u\|_{L^\infty(\Sigma)}.\tag{3.6}$$

Therefore, representation (3.4) and estimates (1.5), (3.5), (3.6) imply

$$|\sigma(t)| \leqslant \int_0^t C\,|\sigma(s)|\,\mathrm{d}s + C\,\|\partial_t\partial_\nu u\|_{L^\infty(\Sigma)}.$$

Here and henceforth, $C>0$ is a generic constant depending only on data. Hence, Gronwall's lemma yields

$$|\sigma(t)| \leqslant C\,\|\partial_t\partial_\nu u\|_{L^\infty(\Sigma)}\,e^{Ct} \leqslant Ce^{CT}\,\|\partial_t\partial_\nu u\|_{L^\infty(\Sigma)}\,,\quad t\in(0,T).$$

Then (1.3) follows and the proof is complete. □

Proof of Theorem 2. Set $u=u_1-u_2=u(\sigma_1)-u(\sigma_2)$. Then, according to (H4) and (H6), u is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u-\Delta_x u-q(x,t)u = F\left(t,x,\sigma_1(t),u_2(x,t)\right)-F\left(t,x,\sigma_2(t),u_2(x,t)\right), & (x,t)\in Q,\\ u(x,0)=0, & x\in\Omega,\\ u(t,x)=0, & (t,x)\in\Sigma,\end{cases} \tag{3.7}$$

with

$$q(x,t)=\int_0^1 \partial_u F\left[t,x,\sigma_1(t),u_2(x,t)+\tau(u_1(x,t)-u_2(x,t))\right]\mathrm{d}\tau. \tag{3.8}$$

Note that assumptions (H4) and (H6) imply that $q\in\mathcal{C}^1(\overline{Q})$.

On the other hand, in view of (H4),

$$F\left(t,x,\sigma_1(t),u_2(x,t)\right)-F\left(t,x,\sigma_2(t),u_2(x,t)\right)=(\sigma_1(t)-\sigma_2(t))G(x,t),$$

with

$$G(x,t)=\int_0^1 \partial_\sigma F\left(t,x,\sigma_2(t)+s(\sigma_1(t)-\sigma_2(t)),u_2(x,t)\right)\mathrm{d}s.$$

Using this representation, we deduce that u is the solution of

$$\begin{cases} \partial_t u-\Delta_x u-q(x,t)u=(\sigma_1(t)-\sigma_2(t))G(x,t), & (x,t)\in Q,\\ u(x,0)=0, & x\in\Omega,\\ u(x,t)=0, & (t,x)\in\Sigma.\end{cases} \tag{3.9}$$

Let us remark that (H4) and (H6) imply that $G\in\mathcal{C}^{2,1}(\overline{Q})$. Let $U(t,x;s,y)$ be the fundamental solution of (2.1) with $q(x,t)$ defined by (3.8). Then, according to Theorem 9.1 of [It], for $\sigma(t)=\sigma_1(t)-\sigma_2(t)$, we have the representation

$$u(x,t)=\int_0^t\int_\Omega U(x,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\mathrm{d}s+\int_0^t\int_\Gamma U(x,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s.$$

Since $u(x,t)=0$, $(t,x)\in\Sigma$, we obtain

$$\int_0^t\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\mathrm{d}s=-\int_0^t\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s, \tag{3.10}$$

with x_0 defined in assumption (H5). Combining Lemma 1 and Lemma 2 with some arguments used in the proof of Theorem 1, we prove that f_1 and f_2 defined respectively by

$$f_1(t)=\int_0^t\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}yds,$$

$$f_2(t)=\int_0^t\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\mathrm{d}s,$$

admit a derivative with respect to t and

$$f_1'(t)=\sigma(t)G(x_0,t)+\int_0^t\partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\right)\mathrm{d}s,$$

$$f_2'(t)=\int_0^t \partial_t\left(\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\right)\mathrm{d}s,$$

$$\begin{aligned}&\left|\int_0^t \partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\right)\mathrm{d}s\right|\\ &\qquad\leqslant C\int_0^t |\sigma(s)|\,\|G(\cdot,s)\|_{\mathcal{C}_x^2(\overline{\Omega})}\,\mathrm{d}s\leqslant C\int_0^t|\sigma(s)|\,\mathrm{d}s\end{aligned}\tag{3.11}$$

and

$$\left|\int_0^t \partial_t\left(\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\right)\mathrm{d}s\right|\leqslant C\left\|\partial_t\partial_\nu u\right\|_{L^\infty(\Sigma)}.\tag{3.12}$$

Here and in the sequel $C>0$ is a generic constant that can depend only on data.

Taking the t-derivative of both sides of identity (3.10), we obtain

$$\begin{aligned}\sigma(t)G(x_0,t)=&-\int_0^t \partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\right)\mathrm{d}s\\ &-\int_0^t \partial_t\left(\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\right)\mathrm{d}s.\end{aligned}$$

Let us observe that (H5) and $\max(\|\sigma_1\|_\infty,\|\sigma_2\|_\infty)\leqslant M$ imply

$$\begin{aligned}|G(x_0,t)|&=\int_0^1 \left|\partial_\sigma F\left(t,x_0,\sigma_2(t)+s(\sigma_1(t)-\sigma_2(t)),g(x_0,t)\right)\right|\mathrm{d}s\\ &\geqslant \inf_{t\in[0,T],\sigma\in[-M,M]}|\partial_\sigma F(t,x_0,\sigma,g(x_0,t))|>0.\end{aligned}\tag{3.13}$$

Then,

$$\begin{aligned}\sigma(t)=&H(t)\int_0^t \partial_t\left(\int_\Omega U(x_0,t;y,s)\sigma(s)G(y,s)\mathrm{d}y\right)\mathrm{d}s\\ &+H(t)\int_0^t \partial_t\left(\int_\Gamma U(x_0,t;y,s)\partial_\nu u(y,s)\mathrm{d}\sigma(y)\right)\mathrm{d}s,\end{aligned}$$

where $H(t)=-1/G(x_0,t)$. Hence, (3.11), (3.12) and (3.13) imply

$$|\sigma(t)|\leqslant\int_0^t C\,|\sigma(s)|\,\mathrm{d}s+C\left\|\partial_t\partial_\nu u\right\|_{L^\infty(\Sigma)}.$$

We complete the proof of Theorem 2 by applying Gronwall's lemma. □

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